NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

2024 Year 12 Course Assessment Task 4 (Trial Examination)
Monday, 19 August 2024

General instructions

- Working time – 3 hours. (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 13)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12MXX.1 – Ms Ham ◯ 12MXX.2 – Mr Ho ◯ 12MXX.3 – Mr Lam

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	%
MARKS	/10	/16	/14	/16	/16	/15	/13	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 13).

Glossary

- $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ – set of all natural numbers
- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3\}$ – set of all integers.
- $\mathbb{Z}^+$ – set of all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers
- $\mathbb{C}$ – set of all complex numbers

Questions **Marks**

1. The polynomial $P(z)$ has real coefficients, and $z = 2 + i$ is a root of $P(z)$. **1**

Which quadratic polynomial must be a factor of $P(z)$?

(A) $z^2 - 4z + 5$ (C) $z^2 - 4z + 3$

(B) $z^2 + 4z + 5$ (D) $z^2 + 4z + 3$

2. A particle starts from rest and moves in a way such that its acceleration a in metres per second per second is given by **1**

$$a = 1 + v$$

where v metres per second, is the velocity of the particle after t seconds.
What is the velocity of the particle after $\log_e(e + 1)$ seconds?

(A) e (C) $e^2 + 1$

(B) $e + 1$ (D) $\log_e(1 + e) + 1$

3. It is given that **1**

$$\begin{aligned} \underset{\sim}{a} &= \underset{\sim}{i} + \underset{\sim}{j} \\ \underset{\sim}{b} &= \underset{\sim}{i} - \underset{\sim}{j} \quad \text{and} \\ \underset{\sim}{c} &= \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k} \end{aligned}$$

If $\underset{\sim}{v}$ is a unit vector such that $\underset{\sim}{a} \cdot \underset{\sim}{v} = 0$ and $\underset{\sim}{b} \cdot \underset{\sim}{v} = 0$, which of the following values is equal to $\left|\underset{\sim}{c} \cdot \underset{\sim}{v}\right|$?

(A) 0 (B) 1 (C) 2 (D) 3

4. Given $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are unit vectors, if $\underset{\sim}{a} + \underset{\sim}{b}$ results in a unit vector, which of the following is $|\underset{\sim}{a} - \underset{\sim}{b}|$? **1**

(A) $\sqrt{2}$ (B) $\dfrac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $\dfrac{1}{\sqrt{3}}$

5. Consider the statement: **1**

If I have solar panels on my rooftop and use public transport, then I am reducing my carbon emissions

Which of the following is the *contrapositive* of the statement?

(A) If I am reducing my carbon emissions, then I have solar panels on my rooftop and take public transport.

(B) If I am not reducing my carbon emissions, then I do not have solar panels on my rooftop or I do not use public transport.

(C) If I am reducing my carbon emissions, then I do not have solar panels on my rooftop and I do not use public transport.

(D) If I am not reducing my carbon emissions, then I do not have solar panels on my rooftop but use public transport.

6. Consider the complex numbers z such that **1**

$$z = \cos^2\theta + i\sin^2\theta \quad \text{with } \theta \in \left[0, \frac{\pi}{2}\right]$$

What is the range of possible values of $|z|$?

(A) $[0, 1]$

(B) $\left[\dfrac{1}{\sqrt{2}}, 1\right]$

(C) $\left[\dfrac{1}{2}, 1\right]$

(D) $\left[\dfrac{1}{4}, 1\right]$

7. If $z = x + iy$, where x and $y \in \mathbb{R}$ and $z \in \mathbb{C}$, which of the following **does not** intersect the graph given by $|z - 5| = 2$? **1**

(A) $\operatorname{Arg}(z-3) = \dfrac{\pi}{2}$

(B) $\operatorname{Im}(z) = 2$

(C) $2\operatorname{Re}(z) + 3\operatorname{Im}(z) = 6$

(D) $|z - 5 - 5i| = 4$

8. Which of the following is an expression for $\displaystyle\int \cos^3 x\, dx$? **1**

(A) $\sin x \cos^2 x + \dfrac{1}{3}\sin^3 x + C$

(B) $\sin x \cos^2 x - \dfrac{1}{3}\sin^3 x + C$

(C) $\sin x + \dfrac{1}{3}\sin^3 x + C$

(D) $\sin x - \dfrac{1}{3}\sin^3 x + C$

9. What does $\displaystyle\int_0^{\ln 2} \frac{1}{e^x+1}\,dx$ transform into by using the substitution $u = e^x + 1$? **1**

(A) $\displaystyle\int_2^3 \left(\frac{1}{u} - \frac{1}{u-1}\right) du$

(B) $\displaystyle\int_2^3 \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

(C) $\displaystyle\int_1^3 \left(\frac{1}{u} - \frac{1}{u-1}\right) du$

(D) $\displaystyle\int_2^{e^2+1} \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

10. A particle is moving in simple harmonic motion. The displacement of the particle is x and its velocity v, is given by the equation $v^2 = n^2\left(2kx - x^2\right)$, where n and k are constants. The particle is initially at $x = k$. **1**

Which function, in terms of time t, could represent the motion of the particle?

(A) $x = k\cos(nt)$

(B) $x = k\sin(nt) + k$

(C) $x = 2k\cos(nt) - k$

(D) $x = 2k\sin(nt) + k$

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) **Marks**

(a) i. Find the square roots of $3 + 4i$. **3**

ii. Hence or otherwise, solve **2**

$$z^2 + iz - 1 - i = 0$$

(b) Shade the region on the Argand diagram where the following inequalities hold simultaneously: **3**

$$|z - 2 - 2i| \leq 2$$
$$|\mathrm{Im}(z - 2i)| \geq 1$$

(c) Let ω be a non-real cube root of unity.

i. Show that $1 + \omega + \omega^2 = 0$. **1**

ii. Hence or otherwise, show that **2**

$$\omega\left(1 + 2\omega + 3\omega^2\right)^2 = -3$$

(d) A particle moves in a straight line. Its displacement x metres from the origin after t seconds is given by

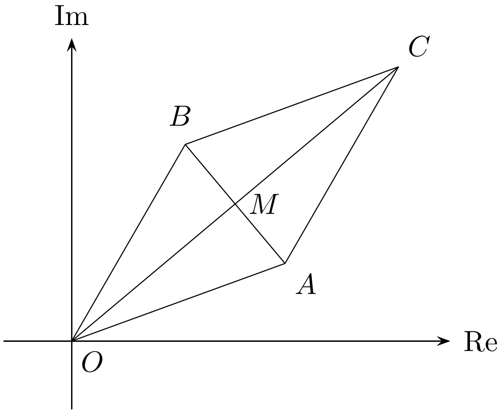
$$x = \frac{1}{\sqrt{3}}\cos 2t + \sin 2t - \frac{1}{\sqrt{3}}$$

i. Prove the particle is moving in simple harmonic motion, centred at $x = -\dfrac{1}{\sqrt{3}}$. **2**

ii. Find the first three times *after* $t = 0$, where the particle has a **speed** of $2\,\mathrm{ms}^{-1}$. **3**

Question 12 (14 marks) **Marks**

(a) Let $z_1 = \cos\alpha + i\sin\alpha$ and $z_2 = \cos\beta + i\sin\beta$ where $0 < \alpha < \beta < \frac{\pi}{2}$. The complex numbers z_1, z_2 and $(z_1 + z_2)$ are represented by the points A, B and C respectively in the Argand diagram.



Let $z_1 + z_2 = r(\cos\theta + i\sin\theta)$, with OC and AB intersecting at the point M.

i. Give a brief reason why $OACB$ is a rhombus. **1**

ii. Show that **2**

$$r = 2\cos\left(\frac{\beta - \alpha}{2}\right)$$

iii. Show that $\theta = \frac{1}{2}(\alpha + \beta)$, and hence show that **2**

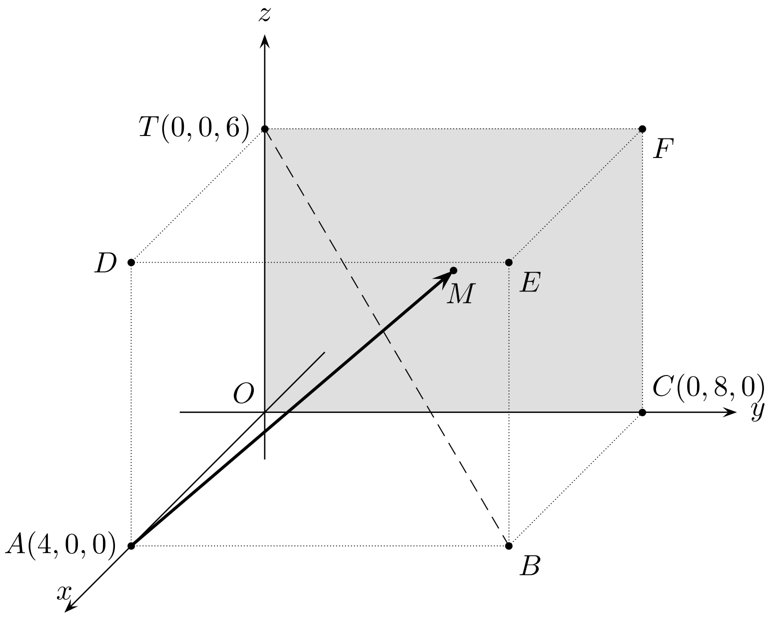
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\beta - \alpha}{2}\right)\cos\left(\frac{\beta + \alpha}{2}\right)$$

(b) Suppose $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are distinct, non-zero vectors with the property **2**

$$\operatorname{proj}_{\underset{\sim}{w}} \underset{\sim}{u} = \operatorname{proj}_{\underset{\sim}{w}} \underset{\sim}{v}$$

Prove that $\left(\underset{\sim}{u} - \underset{\sim}{v}\right)$ is perpendicular to $\underset{\sim}{w}$.

(c) A rectangular prism is defined with vertices $A(4, 0, 0)$, $C(0, 8, 0)$ and $T(0, 0, 6)$. Point M is the centre of the plane with vertices $OCFT$ and M has coordinates $(0, 4, 3)$.



i. Determine the vector equation of the line BT, which is one of the diagonals of the prism. **2**

ii. Find the the vector equation of the sphere that contains all of the vertices of the rectangular prism, in the form $\left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r$. **2**

Hint: Consider the midpoint of the diagonal BT.

iii. Prove using vector methods, that the lines AM and BT are skew. **3**

Examination continues overleaf…

Question 13 (16 marks) **Marks**

(a) The function $f(x) = x^x$ is defined and positive for all $x > 0$. **3**

By differentiating $\ln\Big(f(x)\Big)$, find the value of x at which $f(x)$ is at its minimum.

(b) i. Find real numbers a, b and c such that **2**

$$\frac{5}{x^2(2-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{2-x}$$

ii. Hence or otherwise, find **2**

$$\int \frac{20}{x^2(2-x)}\,dx$$

(c) Find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x - 2\sin x}$. **4**

(d) Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx$, for all integers $n \geq 0$.

i. Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$, for all integers $n \geq 2$. **3**

ii. Hence find **2**

$$\int_0^{\frac{\pi}{4}} \tan^5 x\,dx$$

Question 14 (16 marks) **Marks**

(a) i. Using a suitable substitution, show that **1**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

ii. A function $f(x)$ has the property such that $f(x) + f(a-x) = f(a)$. **2**

Using part (i) or otherwise, show that

$$\int_0^a f(x)\,dx = \frac{a}{2}f(a)$$

(b) If z and w are complex numbers such that $|z| = |w|$, show that **2**

$$\left(\frac{1}{2}(z+w)\right)\left(\overline{\frac{1}{2}(z+w)}\right) + \left(\frac{1}{2}(z-w)\right)\left(\overline{\frac{1}{2}(z-w)}\right) = z\overline{z}$$

(c) Let $z = \cos\theta + i\sin\theta$.

i. Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. **1**

ii. Hence or otherwise, solve $3z^4 - z^3 + 4z^2 - z + 3 = 0$ **3**

(d) i. By using De Moivre's Theorem, show that **2**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

ii. Hence or otherwise, find all solutions to the equation **3**

$$32x^5 - 40x^3 + 10x - 1 = 0$$

iii. Deduce that **2**

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{16}$$

Examination continues overleaf...

Question 15 (15 marks) **Marks**

(a) i. Prove that $\sqrt{14}$ is irrational. **3**

ii. Hence prove it is not possible for both $\sqrt{2n}$ and $\sqrt{7n}$ to be rational for any positive integer n. **2**

(b) A sequence of numbers a_n is given by

$$a_1 = 1 \qquad a_{n+1} = \frac{4 + a_n}{1 + a_n}$$

for $n \geq 1$.

i. Prove by mathematical induction for $n \geq 1$ and $b = -\frac{1}{3}$, **4**

$$a_n = 2\left(\frac{1 + b^n}{1 - b^n}\right)$$

ii. Hence find the limiting value of a_n as $n \to \infty$. **1**

(c) It is given that for any two positive real numbers x and y,

$$\frac{x + y}{2} \geq \sqrt{xy}$$

It is also given that a, b and c are positive real numbers.

i. Use the result above to prove $(a + b + c)^3 \geq 27abc$. **3**

ii. Hence or otherwise, if $abc = 1$, prove that **2**

$$a^5c^4 + b^5a^4 + c^5b^4 \geq 3$$

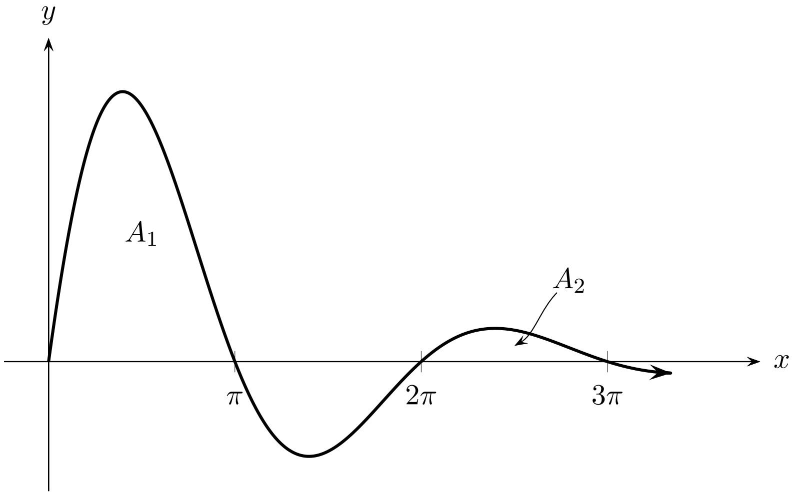
Question 16 (13 marks) **Marks**

(a) A particle moves in a straight line such that at time t, its displacement from the origin is x, and velocity v. **3**

Find x as a function of time given $\ddot{x} = -2e^{-x}$, and when $t = 0$, $x = 0$ and $v = 2$.

(b) The diagram shows a sketch of part of the curve with equation

$$y = e^{-x} \sin x \quad x \geq 0$$



i. Show that **3**

$$\int e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x} (\sin x + \cos x) + C$$

ii. The terms A_1, A_2, $\cdots$, A_n represent successive areas above the x axis and bounded by the curve $y = e^{-x} \sin x$. **2**

The area A_n is bounded by the curve $y = e^{-x} \sin x$ between $x = (2n-2)\pi$ and $x = (2n-1)\pi$.

The areas represented by A_1 and A_2 are shown in the diagram.

Show that

$$A_n = \frac{1}{2} \left(e^{(1-2n)\pi} + e^{(2-2n)\pi} \right)$$

iii. Show that **3**

$$A_1 + A_2 + A_3 + \cdots$$

is a geometric series, and has $\dfrac{e^{\pi}}{2(e^{\pi} - 1)}$ as its limiting sum.

iv. Given that $\displaystyle\lim_{n \to \infty} \int_0^n e^{-x} \sin x \, dx = \frac{1}{2}$, find the exact value of **2**

$$\lim_{n \to \infty} \int_0^n \left| e^{-x} \sin x \right| \, dx$$

End of paper.

Sample Band E4 Responses

1. (A) **2.** (A) **3.** (D) **4.** (C) **5.** (B)
6. (B) **7.** (A) **8.** (D) **9.** (B) **10.** (B)

Note: Working out is not marked for MCQ, though solutions are provided as a learning opportunity on mathematical writing.

Question 1

One of the roots to $P(z) = 0$ is $\alpha = 2 + i$. Hence, one of the other roots is $\beta = \overline{\alpha} = 2 - i$

$$\begin{aligned}\text{Sum of roots } &= \alpha + \beta \\ &= (2+i) + (2-i) \\ &= 4 \\ \text{Product of roots } &= \alpha\beta \\ &= (2+i)(2-i) \\ &= 5\end{aligned}$$

Hence, the quadratic factor from this pair of conjugate roots will be

$$z^2 - (\alpha + \beta)z + \alpha\beta = z^2 - 4z + 5$$

Correct option: (A)

Question 2

Rewriting a as $\dfrac{dv}{dt}$ and solving the differential equation:

$$\begin{aligned}\frac{dv}{dt} &= 1 + v \\ \frac{dv}{1+v} &= dt\end{aligned}$$

Integrating both sides,

$$\begin{aligned}\int \frac{dv}{1+v} &= \int dt \\ \log_e(1+v) &= t + C_1 \\ 1 + v &= e^{t+C_1} \\ \therefore v &= Ae^t - 1\end{aligned}$$

When $t = 0$, $v = 0$ (starts from rest):

$$\begin{aligned}0 &= Ae^0 - 1 \\ \therefore A &= 1 \\ v &= e^1 - 1\end{aligned}$$

When $t = \log_e(e+1)$,

$$\begin{aligned}v &= e^{\log_e(e+1)} - 1 \\ &= e + 1 - 1 \\ &= e\end{aligned}$$

Correct option: (A)

Question 3

Rewriting in column vector notation:

$$\underset{\sim}{a} = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \underset{\sim}{b} = \begin{pmatrix}1\\-1\\0\end{pmatrix} \qquad \underset{\sim}{c} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

Also, let

$$\underset{\sim}{v} = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

where x, y and $z \in \mathbb{R}$. Now examine the various dot products:

$$\underset{\sim}{a} \cdot \underset{\sim}{v} = 0$$

i.e.

$$\begin{aligned}\frac{1}{\sqrt{x^2+y^2+z^2}}\Big(1(x) + 1(y) + 0(z)\Big) &= 0 \\ \therefore x + y &= 0 \\ x &= -y \qquad (1)\end{aligned}$$

Similarly,

$$\underset{\sim}{b} \cdot \underset{\sim}{v} = 0$$

i.e.

$$\begin{aligned}\frac{1}{\sqrt{x^2+y^2+z^2}}\Big(1(x) - 1(y) + 0(z)\Big) &= 0 \\ \therefore x - y &= 0 \\ x &= y \qquad (2)\end{aligned}$$

Adding (1) and (2):

$$\begin{aligned}2x &= 0 \\ x &= 0 \\ \therefore y &= 0\end{aligned}$$

Examine $\underset{\sim}{c} \cdot \underset{\sim}{v}$:

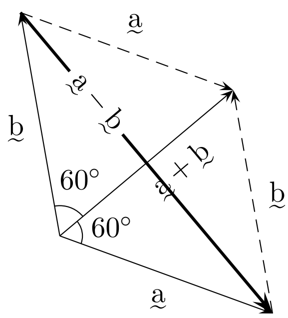
$$\begin{aligned}\underset{\sim}{c} \cdot \underset{\sim}{v} &= \cancel{\frac{1}{\sqrt{0+0+z^2}}}\Big(1(0) + 1(0) + 3\cancel{(z)}\Big) \\ &= 3 \\ \therefore \left|\underset{\sim}{c} \cdot \underset{\sim}{v}\right| &= 3\end{aligned}$$

Correct option: (D)

Question 4

Draw a diagram that represents the situation.

- Given $|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{a} + \underset{\sim}{b}| = 1$, which means $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{a} + \underset{\sim}{b}$ forms an equilateral triangle of side length 1.
- Note that $\underset{\sim}{a} + \underset{\sim}{b}$ forms one of the diagonals of a parallelogram when two vectors are added, and $\underset{\sim}{a} - \underset{\sim}{b}$ is the other diagonal (starting at the arrowhead of vector $\underset{\sim}{b}$, ending at the arrowhead of $\underset{\sim}{a}$):



Using the cosine rule on the triangle formed by $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$:

$$\begin{aligned}
|\underset{\sim}{a} - \underset{\sim}{b}|^2 &= 1^2 + 1^2 - 2(1)(1)\cos 120^\circ \\
&= 1 - 2\left(-\frac{1}{2}\right) \\
&= 3 \\
\therefore |\underset{\sim}{a} - \underset{\sim}{b}| &= \sqrt{3}
\end{aligned}$$

Correct option: (C)

Question 5

Let

- p be the proposition *I have solar panels on my rooftop*
- q be the proposition *I use public transport*
- r be the proposition *I am reducing carbon emissions*

The statement presented is in the form

$$(p \wedge q) \Rightarrow r$$

The contrapositive is

$$\begin{aligned}
\neg r &\Rightarrow \neg(p \wedge q) \\
\neg r &\Rightarrow \neg p \vee \neg q
\end{aligned}$$

...which will read

> *If I am* **not** *reducing my carbon emissions, then I* **do not** *have solar panels* **or** *I* **do not** *use public transport*

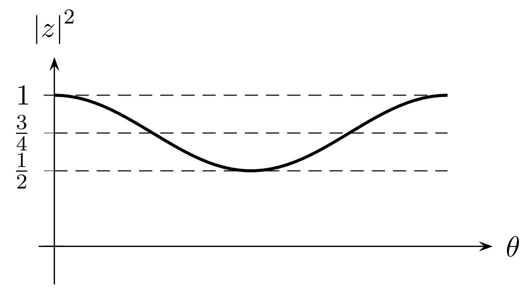
Correct option: (B)

Question 6

Examining $|z|^2 = z\overline{z}$, where $z = \cos^2\theta + i\sin^2\theta$:

$$\begin{aligned}
|z|^2 &= z\overline{z} \\
&= \left(\cos^2\theta + i\sin^2\theta\right)\left(\cos^2\theta - i\sin^2\theta\right) \\
&= \cos^4\theta + \sin^4\theta \\
&= \cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta \\
&= \left(\cos^2\theta + \sin^2\theta\right)^2 - \frac{1}{2}\sin^2 2\theta \\
&= 1 - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) \\
&= 1 - \frac{1}{4} + \frac{1}{4}\cos 4\theta \\
&= \frac{3}{4} + \frac{1}{4}\cos 4\theta
\end{aligned}$$

Sketching one period of $\frac{3}{4} + \frac{1}{4}\cos 4\theta$:



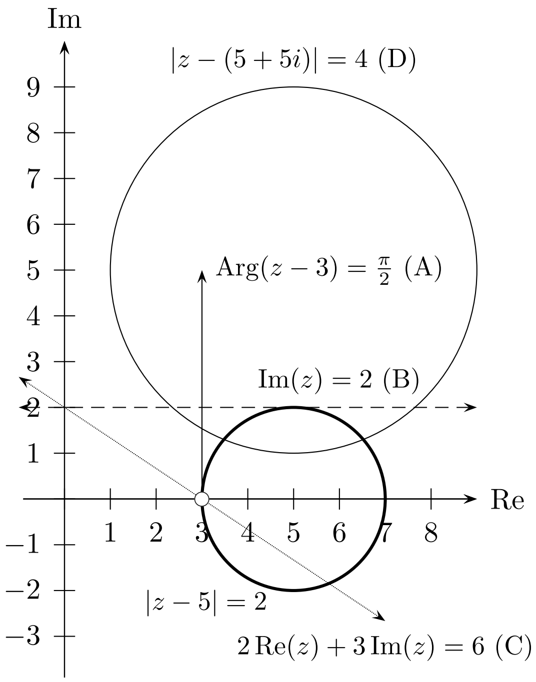
From the graph of θ against $|z|^2$, it's evident that

$$\begin{aligned}
\frac{1}{2} &\le |z|^2 \le 1 \\
\therefore \frac{1}{\sqrt{2}} &\le |z| \le 1
\end{aligned}$$

Correct option: (B)

Question 7

Draw various diagrams to represent the options presented.



(A) $\text{Arg}(z-3)=\dfrac{\pi}{2}$ is the vertical ray commencing at $z=3+0i$, but not inclusive at $z=3+0i$, *which just misses the circle!*

(B) As $\text{Im}(z)=y$, then $\text{Im}(z)=2$ is the horizontal line $y=2$. Tangent to the circle at $z=5+2i$

(C) As $\text{Re}(z)=x$ and $\text{Im}(z)=y$, then

$$\begin{aligned}2\,\text{Re}(z)+3\,\text{Im}(z)&=6\\2x+3y&=6\end{aligned}$$

which crosses the horizontal axis at $z=3$ and vertical axis at $z=2i$; subsequently intersecting the original circle twice.

(D) $|z-(5+5i)|=4$ is a circle of radius 4, centred at $z=5+5i$. Will cross the existing circle twice.

Correct option: (A)

Question 8

Break off one of the $\cos^2 x$ terms and converting to $1-\sin^2 x$:

$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x\cos x\,dx\\&=\int\left(1-\sin^2 x\right)\cos x\,dx\\&=\int\left(\cos x-\sin^2 x\cos x\right)\,dx\\&=\sin x-\frac{1}{3}\sin^3 x+C\end{aligned}$$

Correct option: (D)

Question 9

Calculating the transformations:

$$\begin{aligned}u=e^x+1 &\quad\Rightarrow\quad e^x=u-1\\\frac{du}{dx}=e^x &\quad\Rightarrow\quad \frac{du}{e^x}=dx\\&\quad\frac{du}{u-1}=dx\end{aligned}$$

Transforming the limits:

$$\begin{aligned}x=0 &\qquad u=e^0+1=2\\x=\ln 2 &\qquad u=e^{\ln 2}+1=3\end{aligned}$$

Rewriting the integral in terms of u:

$$\begin{aligned}\int_0^{\ln 2}\frac{1}{e^x+1}\,dx&=\int_{u=2}^{u=3}\frac{1}{u}\times\frac{du}{u-1}\\&=\int_2^3\frac{du}{u(u-1)}\end{aligned}$$

Decomposing $\dfrac{1}{u(u-1)}$ into partial fractions, where A and $B\in\mathbb{R}$:

$$\begin{aligned}\frac{1}{u(u-1)}&\equiv\frac{A}{u}+\frac{B}{u-1}\\1&\equiv A(u-1)+Bu\end{aligned}$$

Let $u=0$:

$$\begin{aligned}1&=A(-1)\\\therefore A&=-1\end{aligned}$$

Let $u=1$:

$$\begin{aligned}1&=0+B\\\therefore B&=1\end{aligned}$$

$$\therefore\frac{1}{u(u-1)}\equiv\frac{1}{u-1}-\frac{1}{u}$$

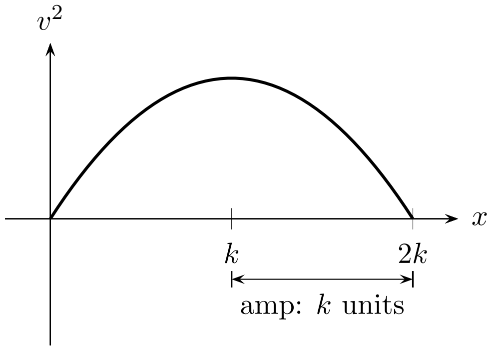
Correct option: (B)

Question 10

Sketch the graph of v^2 against x, being a concave down parabola:

$$v^2 = n^2\left(2kx - x^2\right) = n^2\Big(x(2k-x)\Big)$$

– Graph has roots at $x = 0$ and $x = 2k$
– $v^2 \geq 0$



– The centre of motion is at $x_0 = k$
– From the graph, the amplitude of the motion is also k units.
– The graph commences from the centre of motion $x = k$.

Hence $x = k\sin(nt) + k$.

Correct option: (B)

Question 11 (Lam)

(a) i. (3 marks)

✓ [1] for some attempt at equating real and imaginary parts.

✓ [2] for substantial progress to reach a single quartic in terms of x or equivalent merit.

✓ [3] for fully correct solution

Let $z = x + iy$, such that

$$\begin{aligned} z^2 &= 3+4i \\ (x+iy)^2 &= 3+4i \\ \left(x^2-y^2\right) + i(2xy) &= 3+4i \end{aligned}$$

Equating real and imaginary parts,

$$\begin{cases} x^2 - y^2 = 3 & (1) \\ 2xy = 4 & (2) \end{cases}$$

Find y in terms of x via (2):

$$\begin{aligned} 2xy &= 4 \\ xy &= 2 \\ y &= \frac{2}{x} \text{ (substitute into (1))} \\ \underbrace{x^2 - \left(\frac{2}{x}\right)^2}_{\times x^2} &= \underset{\times x^2}{3} \\ x^4 - 4 &= 3x^2 \\ x^4 - 3x^2 - 4 &= 0 \\ \left(x^2-4\right)\left(x^2+1\right) &= 0 \end{aligned}$$

As $x \in \mathbb{R}$,

$$\begin{aligned} x^2 - 4 &= 0 \\ x &= \pm 2 \\ y &= \frac{2}{\pm 2} = \pm 1 \\ \therefore z &= \pm 2 \pm i = \pm(2+i) \end{aligned}$$

ii. (2 marks)

✓ [1] for obtaining $z = \dfrac{-i \pm \sqrt{3+4i}}{2}$ with working out, or equivalent merit.

✓ [2] for fully correct roots of $z = 1$ or $z = -1 - i$.

$$z^2 + iz - (1+i) = 0$$

Applying the quadratic formula,

$$\begin{aligned} z &= \frac{-i \pm \sqrt{i^2 + 4(1)(1+i)}}{2} \\ &= \frac{-i \pm \sqrt{3+4i}}{2} \\ &= \frac{-i \pm (2+i)}{2} \\ &= \frac{-i+2+i}{2} \text{ or } \frac{-i-2-i}{2} \\ &= 1 \text{ or } -1-i \end{aligned}$$

(b) (3 marks)

✓ [1] for correctly sketching one of either the circle or horizontal line

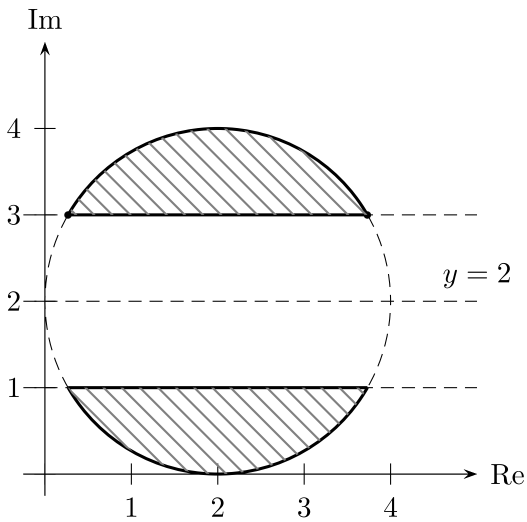
✓ [2] for correctly sketching both the circle or horizontal line

✓ [3] for correct shading including the circle, horizontal line and also non-inclusive boundaries

- $|z-(2+2i)|=2$ is a circle centred at $2+2i$, radius of 2.
  $\therefore$ $|z-(2+2i|\le 2$ indicates it's the area inside the circle.
- $|\text{Im}(z-2i)|\ge 1$: letting $z=x+iy$,

$$\begin{gathered}|\text{Im}(x+iy-2i)|\ge 1\\ |\text{Im}(x+i(y-2))|\ge 1\\ \therefore |y-2|\ge 1\\ y-2\ge 1 \text{ or } y-2\le -1\\ y\ge 3 \text{ or } y\le 1\end{gathered}$$

This set of a points in the Argand diagram is at least 1 unit away from the line $y=2$.



(c) i. (1 mark)

$$\begin{gathered}\omega^3=1\\ \therefore \omega^3-1=0\\ (\omega-1)\left(\omega^2+\omega+1\right)=0\end{gathered}$$

As $\omega\notin\mathbb{R}$

$$\therefore 1+\omega+\omega^2=0$$

ii. (2 marks)

✓ [1] for some usage of the result in (i), but could not reach the required result.

✓ [2] for fully correct solution.

Some different ways of expressing the result from (i):

- $1+\omega+\omega^2=0$
- $1=-\omega-\omega^2$

$$\begin{aligned}&\omega\left(1+2\omega+3\omega^2\right)^2\\ &=\omega\Big(\left(-\omega-\omega^2\right)+2\omega+3\omega^2\Big)^2\\ &=\omega\Big(\omega+2\omega^2\Big)^2\\ &=\omega\times\omega^2\left(1+2\omega\right)^2\\ &=\cancelto{1}{\omega^3}\left(1+4\omega+4\omega^2\right)\\ &=\cancelto{0}{4+4\omega+4\omega^2}-3\\ &=-3\end{aligned}$$

(d) i. (2 marks)

✓ [1] correctly differentiates to obtain the second derivative.

✓ [2] correctly finds the second derivative with respect to time, masks the time with displacement term to show the required result.

Note The auxiliary angle method is used here instead of part (ii) to collapse the two sinusoids into a single sinusoid in order to allow for quicker differentiation.

$$\begin{aligned}&\frac{1}{\sqrt{3}}\cos 2t+\sin 2t\\ &\equiv R\cos(2t-\alpha)\\ &=R\cos 2t\cos\alpha+R\sin 2t\sin\alpha\end{aligned}$$

Equating coefficients,

$$\begin{cases}R\cos\alpha=\frac{1}{\sqrt{3}} & (1)\\ R\sin\alpha=1 & (2)\end{cases}$$

Dividing (2) by (1):

$$\begin{gathered}\tan\alpha=\frac{1}{\frac{1}{\sqrt{3}}}=\sqrt{3}\\ \therefore \alpha=\frac{\pi}{3}\end{gathered}$$

Substituting result into (2):

$$\begin{aligned}
R\sin\frac{\pi}{3} &= 1\\
R\times\frac{\sqrt{3}}{2} &= 1\\
\therefore R &= \frac{2}{\sqrt{3}}\\
\therefore x &= \frac{2}{\sqrt{3}}\cos\left(2t-\frac{\pi}{3}\right)-\frac{1}{\sqrt{3}}\\
\dot{x} &= \frac{2}{\sqrt{3}}\times -2\sin\left(2t-\frac{\pi}{3}\right)\\
\ddot{x} &= \frac{2}{\sqrt{3}}\times -2^2\cos\left(2t-\frac{\pi}{3}\right)\\
&= -2^2\left(\frac{2}{\sqrt{3}}\cos\left(2t-\frac{\pi}{3}\right)\right)\\
&= -2^2\left(x+\frac{1}{\sqrt{3}}\right)\\
&\equiv -n^2\left(x-x_0\right)
\end{aligned}$$

where $n=2$ and $x_0=-\frac{1}{\sqrt{3}}$. Hence acceleration is proportional to, but directed against the displacement from the centre of motion $x=-\frac{1}{\sqrt{3}}$. Hence the particle is undergoing simple harmonic motion.

ii. (3 marks)

- ✓ [1] for some attempting to solve $\dot{x}=\pm 2$. (Or, if the auxiliary angle method was not used in part (i), then finding the equivalent R and α values)
- ✓ [2] for correctly finding some of the roots to $\dot{x}=\pm 2$.
- ✓ [3] for fully correct solution.

$$\begin{aligned}
\dot{x} &= \frac{2}{\sqrt{3}}\times -2\sin\left(2t-\frac{\pi}{3}\right)\\
&= -\frac{4}{\sqrt{3}}\sin\left(2t-\frac{\pi}{3}\right)
\end{aligned}$$

Solving for $\dot{x}=\pm 2$,

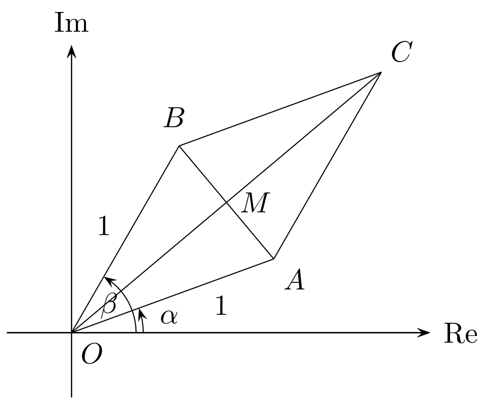
$$\begin{aligned}
-\frac{4}{\sqrt{3}}\sin\left(2t-\frac{\pi}{3}\right) &= \pm 2\\
-\frac{2}{\sqrt{3}}\sin\left(2t-\frac{\pi}{3}\right) &= \pm 1\\
\sin\left(2t-\frac{\pi}{3}\right) &= \pm\frac{\sqrt{3}}{2}\\
2t-\frac{\pi}{3} &= -\frac{\pi}{3},\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\\
2t &= \emptyset,\frac{2\pi}{3},\pi,\frac{5\pi}{3}\\
t &= \frac{\pi}{3},\frac{\pi}{2},\frac{5\pi}{6}
\end{aligned}$$

Question 12 (Lam)

(a) i. (1 mark)
$OACB$ is a parallelogram with one pair of adjacent sides equal.

ii. (2 marks)

- ✓ [1] for some algebraic manipulation to demonstrate the result, but without necessary reasoning; or to obtain only $OC^2 = 2+2\cos(\beta-\alpha)$; or equivalent merit.
- ✓ [2] for fully correct solution.



- In $\triangle OAB$, $\angle AOB=\beta-\alpha$.
- Hence $\angle OAC = 180^\circ-(\beta-\alpha)$
- Applying the cosine rule in $\triangle AOC$:

$$\begin{aligned}
OC^2 &= OA^2 + AC^2 \\
&\quad - 2(OA)(AC)\cos\angle OAC \\
&= 1^2 + 1^2 \\
&\quad - 2\cos\Big(180^\circ - (\beta - \alpha)\Big) \\
&= 2 + 2\cos(\beta - \alpha) \\
&= 4\cos^2\left(\frac{\beta - \alpha}{2}\right) \\
\therefore OC &= 2\cos\left(\frac{\beta - \alpha}{2}\right) \\
r &= 2\cos\left(\frac{\beta - \alpha}{2}\right)
\end{aligned}$$

iii. (2 marks)

- ✓ [1] for showing the required value for θ.
- ✓ [2] for fully correct solution.

- $\text{Arg}(z_1 + z_2) = \theta$
- As the diagonals of a rhombus bisect the angle, $\angle AOM = \frac{\beta-\alpha}{2}$.
- As $\alpha + \angle AOM = \theta$,

$$\begin{aligned}
\theta &= \alpha + \angle AOM \\
&= \alpha + \frac{\beta - \alpha}{2} \\
&= \frac{\beta + \alpha}{2} \\
&= \frac{1}{2}(\alpha + \beta)
\end{aligned}$$

- Finally,

$$\begin{aligned}
&z_1 + z_2 \\
&= \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta \\
&= (\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta) \\
&\equiv \overbrace{2\cos\left(\frac{\beta - \alpha}{2}\right)}^{=r}(\cos\theta + i\sin\theta)
\end{aligned}$$

Equating real parts,

$$\begin{aligned}
&\cos\alpha + \cos\beta \\
&= 2\cos\left(\frac{\beta - \alpha}{2}\right)\cos\left(\frac{\beta + \alpha}{2}\right)
\end{aligned}$$

(b) (2 marks)

- ✓ [1] for some use of the projection formula with correct use of vector notation.
- ✓ [2] for fully correct solution.

As $\text{proj}_{\underset{\sim}{w}}\,\underset{\sim}{u} = \text{proj}_{\underset{\sim}{w}}\,\underset{\sim}{v}$,

$$\therefore \frac{\underset{\sim}{u}\cdot\underset{\sim}{w}}{|\underset{\sim}{w}|}\widehat{\underset{\sim}{w}} = \frac{\underset{\sim}{v}\cdot\underset{\sim}{w}}{|\underset{\sim}{w}|}\widehat{\underset{\sim}{w}}$$

Equating the scalar multiples applied to $\widehat{\underset{\sim}{w}}$,

$$\begin{aligned}
\frac{\underset{\sim}{u}\cdot\underset{\sim}{w}}{\cancel{|\underset{\sim}{w}|}} &= \frac{\underset{\sim}{v}\cdot\underset{\sim}{w}}{\cancel{|\underset{\sim}{w}|}} \\
\therefore \underset{\sim}{u}\cdot\underset{\sim}{w} &= \underset{\sim}{v}\cdot\underset{\sim}{w} \\
\underset{\sim}{u}\cdot\underset{\sim}{w} - \underset{\sim}{v}\cdot\underset{\sim}{w} &= 0 \\
\underset{\sim}{w}\cdot(\underset{\sim}{u} - \underset{\sim}{v}) &= 0
\end{aligned}$$

Hence $\underset{\sim}{w}$ is perpendicular to $\underset{\sim}{u} - \underset{\sim}{v}$.

(c) i. (2 marks)

- ✓ [1] for attempts to find a vector in the direction of BT (or TB).
- ✓ [2] for fully correct solution.

B has coordinates $(4, 8, 0)$

$$\begin{aligned}
\underset{\sim}{r}_{BT} &= \begin{pmatrix}0\\0\\6\end{pmatrix} + \lambda\left(\begin{pmatrix}0\\0\\6\end{pmatrix} - \begin{pmatrix}4\\8\\0\end{pmatrix}\right) \\
&= \begin{pmatrix}0\\0\\6\end{pmatrix} + \lambda\begin{pmatrix}-4\\-8\\6\end{pmatrix}
\end{aligned}$$

ii. (2 marks)

- ✓ [1] for finding the midpoint of the diagonal BT, or the radius of the sphere.
- ✓ [2] for fully correct solution.

$$\begin{aligned}
MP_{BT} &= \frac{1}{2}\left(\begin{pmatrix}4\\8\\0\end{pmatrix} + \begin{pmatrix}0\\0\\6\end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix}4\\8\\6\end{pmatrix} \\
&= \begin{pmatrix}2\\4\\3\end{pmatrix}
\end{aligned}$$

Radius will be the distance from MP_{BT} to one of the vertices

$$\begin{aligned} r &= \sqrt{(2-0)^2+(4-0)^2+(3-6)^2} \\ &= \sqrt{4+16+9} \\ &= \sqrt{29} \end{aligned}$$

$$\therefore \left| \underset{\sim}{v} - \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \right| = \sqrt{29}$$

iii. (3 marks)

- ✓ [1] for finding r_{AM}, or equivalent merit.
- ✓ [2] for substantial progress in setting up a system of equations by equating $\underset{\sim}{r}_{AM}$ with $\underset{\sim}{r}_{BT}$.
- ✓ [3] for fully correct solution and justification.

$$\overrightarrow{AM} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix}$$

$$\therefore \underset{\sim}{r}_{AM} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix}$$

Attempting to find the point of intersection between AM and BT: this occurs when $\underset{\sim}{r}_{BT} = \underset{\sim}{r}_{AM}$ for a particular value of λ and μ:

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 8 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} 4-4\mu \\ 4\mu \\ 3\mu \end{pmatrix} = \begin{pmatrix} 4\lambda \\ 8\lambda \\ 6-6\lambda \end{pmatrix}$$

which results in the system of equations such that

$$\begin{cases} 4\lambda + 4\mu = 4 & (1) \\ 8\lambda - 4\mu = 0 & (2) \\ 6\lambda + 3\mu = 6 & (3) \end{cases}$$

Adding (1) and (2):

$$\begin{aligned} 12\lambda &= 4 \\ \lambda &= \frac{1}{3} \end{aligned}$$

Substituting into (3):

$$\begin{aligned} 6\left(\frac{1}{3}\right) + 3\mu &= 6 \\ 2 + 3\mu &= 6 \\ 3\mu &= 4 \\ \mu &= \frac{4}{3} \end{aligned}$$

Substituting back into (1):

$$\begin{aligned} 4\lambda + 4\left(\frac{4}{3}\right) &= 2 \\ 4\lambda &= -\frac{10}{3} \\ \lambda &= -\frac{5}{6} \end{aligned}$$

However, these values of λ are inconsistent across the system of equations. Hence no such λ or μ exists, and the lines $\underset{\sim}{r}_{BT}$ and $\underset{\sim}{r}_{AM}$ do not intersect.

Question 13 (Ho)

(a) (3 marks)

- ✓ [1] for successfully applying the product rule to obtain the derivative to $\ln x^x$ to be $1+\ln x$.
- ✓ [2] for the application of the product rule, as well as the justification of why the minimum to $\ln f(x)$ is also the minimum to $f(x)$.
- ✓ [3] for fully correct solution by testing the point $x = \frac{1}{e}$

$$\begin{aligned} y &= x^x \\ \ln y &= x \ln x \\ \frac{d}{dx}\left(\ln\left(x^x\right)\right) &= \frac{d}{dx}\Big(x \ln x\Big) \end{aligned}$$

Applying the product rule,

$$\left| \begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = \dfrac{1}{x} \end{array} \right.$$

$$\frac{d}{dx}\left(\ln\left(x^x\right)\right) = 1 + \ln x$$

As $\ln f(x)$ is monotonic increasing, the minimum for $f(x)$ will occur also at the same location as $\ln f(x)$, i.e. where $\frac{d}{dx}\left(\ln\left(x^x\right)\right) = 0$

$$\begin{aligned} 1 + \ln x &= 0 \\ \ln x &= -1 \\ x &= e^{-1} \end{aligned}$$

Checking for local min:

$$\begin{aligned} \frac{d^2}{dx^2}\left(\ln\left(x^x\right)\right) &= \frac{d}{dx}\left(1 + \ln x\right) \\ &= \left.\frac{1}{x}\right|_{x=\frac{1}{e}} \\ &> 0 \end{aligned}$$

Hence a local min exists at $x = e^{-1}$ for $\ln f(x)$, and the minimum for $f(x)$ also occurs at $x = \frac{1}{e}$

(b) i. (2 marks)

✓ [1] for finding one of a, b or c.

✓ [1] for the fully correct solution.

$$\begin{aligned} \frac{5}{x^2(2-x)} &\equiv \frac{ax+b}{x^2} + \frac{c}{2-x} \\ 5 &\equiv (ax+b)(2-x) + cx^2 \end{aligned}$$

Let $x = 0$,

$$\begin{aligned} 5 &= b(2-0) \\ \therefore b &= \frac{5}{2} \end{aligned}$$

Let $x = 2$,

$$\begin{aligned} 5 &= 0 + c\left(2^2\right) \\ 5 &= 4c \\ c &= \frac{5}{4} \end{aligned}$$

Let $x = 1$,

$$\begin{aligned} 5 &= \left(a(1) + \frac{5}{2}\right) + \frac{5}{4}(1) \\ \frac{15}{4} &= a + \frac{5}{2} \\ a &= \frac{5}{4} \\ \therefore \frac{5}{x^2(2-x)} &\equiv \frac{\frac{5}{4}x + \frac{5}{2}}{x^2} + \frac{\frac{5}{4}}{2-x} \end{aligned}$$

ii. (2 marks)

✓ [1] for correct manipulation to result in $\frac{5}{x} + 10x^{-2} + \frac{5}{2-x}$, or equivalent merit.

✓ [2] for correct solution, including absolute values in logarithms and also $+C$

$$\begin{aligned} &\int \frac{20}{x^2(2-x)}\,dx \\ &= \int 4 \times \left(\frac{\frac{5}{4}x + \frac{5}{2}}{x^2} + \frac{\frac{5}{4}}{2-x}\right)\,dx \\ &= \int \left(\frac{5x+10}{x^2} + \frac{5}{2-x}\right)\,dx \\ &= \int \left(\frac{5}{x} + 10x^{-2} + \frac{5}{2-x}\right)\,dx \\ &= 5\ln|x| - 10x^{-1} - 5\ln|2-x| + C \end{aligned}$$

(c) (4 marks)

- ✓ [1] for correct alteration of limits and converting differential in x into differential in t
- ✓ [2] for further progress to simplify denominator into the simplest quadratic.
- ✓ [3] for further progress and obtaining a denominator that has a perfect square.
- ✓ [4] for fully correct answer.

Using t formulae: let $t = \tan\frac{x}{2}$

$$\begin{aligned} t &= \tan\frac{x}{2} \\ \frac{x}{2} &= \tan^{-1} t \\ x &= 2\tan^{-1} t \\ \therefore \frac{dx}{dt} &= \frac{2}{1+t^2} \\ dx &= \frac{2}{1+t^2}\,dt \end{aligned}$$

Finding the equivalent limits of integration:

$$\left| \begin{array}{ll} x = 0 & t = \tan 0 = 0 \\ x = \frac{\pi}{2} & t = \tan\frac{\pi}{4} = 1 \end{array}\right.$$

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x - 2\sin x} \\ &= \int_{t=0}^{t=1} \frac{\frac{2}{1+t^2}}{3 - \left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right)}\,dt \\ &= \int_0^1 \frac{2}{3\left(1+t^2\right) - \left(1-t^2\right) - 4t}\,dt \\ &= \int_0^1 \frac{2}{3+3t^2-1+t^2-4t}\,dt \\ &= \int_0^1 \frac{2}{4t^2-4t+1+1}\,dt \\ &= \int_0^1 \frac{2}{1+(2t-1)^2}\,dt \\ &= \left[\tan^{-1}(2t-1)\right]_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(-1) \\ &= 2\tan^{-1}(1) \\ &= \frac{\pi}{2} \end{aligned}$$

(d) i. (3 marks)

- ✓ [1] for splitting a $\tan^2 x$ term from the integrand and converting into $\sec^2 x - 1$
- ✓ [2] for the above, as well as splitting the integrand to obtain a term in I_{n-2}.
- ✓ [3] for fully correct solution.

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{4}} \tan^n x\,dx \\ &= \int_0^{\frac{\pi}{4}} \tan^2 x \tan^{n-2} x\,dx \\ &= \int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)\tan^{n-2} x\,dx \\ &= \int_0^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x\,dx \\ &\quad - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx \\ &= \left[\frac{1}{n-1}\tan^{n-1} x\right]_0^{\frac{\pi}{4}} - I_{n-2} \\ &= \frac{1}{n-1}\left(\tan^{n-1}\frac{\pi}{4} - \tan^{n-1} 0\right) \\ &\quad - I_{n-2} \\ &= \frac{1}{n-1} - I_{n-2} \end{aligned}$$

ii. (2 marks)

- ✓ [1] for obtaining the expression $\frac{1}{4} - \left(\frac{1}{2} - I_1\right)$, or equivalent merit.
- ✓ [1] for fully correct solution.

$$\begin{aligned} I_5 &= \int_0^{\frac{\pi}{4}} \tan^5 x\,dx \\ &= \frac{1}{4} - I_3 \\ &= \frac{1}{4} - \left(\frac{1}{2} - I_1\right) \\ &= -\frac{1}{4} + \int_0^{\frac{\pi}{4}} \tan x\,dx \\ &= -\frac{1}{4} + \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx \\ &= -\frac{1}{4} - \Big[\ln|\cos x|\Big]_0^{\frac{\pi}{4}} \\ &= -\frac{1}{4} - \left(\ln\frac{1}{\sqrt{2}} - \ln 1\right) \\ &= -\frac{1}{4} + \frac{1}{2}\ln 2 \end{aligned}$$

Question 14 (Ho)

(a) i. (1 mark)

$$\int_0^a f(x)\,dx$$

Let $u = a - x$, and hence $du = -dx$

$$\left|\begin{array}{ll} x = 0 & u = a \\ x = a & u = 0 \end{array}\right.$$

$$\begin{aligned}\int_0^a f(x)\,dx &= \int_{u=a}^{u=0} f(a-u)\,(-du)\\ &= \int_0^a f(a-u)\,du\\ &= \int_0^a f(a-x)\,dx\end{aligned}$$

by replacing the pronumeral u with x,

$$\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

ii. (2 marks)

✓ [1] for obtaining the primitive of $\int_0^a f(a)\,dx$

✓ [2] for fully correct solution

$$\begin{aligned}\int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx &= \int_0^a f(a)\,dx\\ \int_0^a f(x)\,dx + \int_0^a f(x)\,dx &= \Big[xf(a)\Big]_0^a\\ 2\int_0^a f(x)\,dx &= af(a) - 0f(a)\\ \therefore \int_0^a f(x)\,dx &= \frac{a}{2}f(a)\end{aligned}$$

(b) (2 marks)

✓ [1] significant progress in expanding the expression, or equivalent merit.

✓ [2] for correct final answer.

Note that $z\overline{z} = |z|^2 = w\overline{w} = |w|^2$.

$$\begin{aligned}&\left(\frac{1}{2}(z+w)\right)\left(\overline{\frac{1}{2}(z+w)}\right)\\ &\quad + \left(\frac{1}{2}(z-w)\right)\left(\overline{\frac{1}{2}(z-w)}\right)\\ &= \frac{1}{4}\left((z+w)\left(\overline{z}+\overline{w}\right) + (z-w)\left(\overline{z}-\overline{w}\right)\right)\\ &= \frac{1}{4}\left(z\overline{z} + \cancel{z\overline{w}} + \cancel{w\overline{z}} + w\overline{w}\right.\\ &\quad \left. + z\overline{z} - \cancel{z\overline{w}} - \cancel{w\overline{z}} + w\overline{w}\right)\\ &= \frac{1}{2}\left(z\overline{z} + w\overline{w}\right)\\ &= z\overline{z}\end{aligned}$$

(c) i. (1 mark)

- If $z = \cos\theta + i\sin\theta$, then by De Moivre's Theorem,

$$z^n = \cos(n\theta) + i\sin(n\theta)$$

- Also,

$$\begin{aligned}z^{-n} &= \cos(-n\theta) + i\sin(-n\theta)\\ &= \cos(n\theta) - i\sin(n\theta)\end{aligned}$$

as cosine is an even function, and sine is an odd function.

- Hence,

$$\begin{aligned}z^n + \frac{1}{z^n} &= \cos(n\theta) + i\sin(n\theta)\\ &\quad \cos(n\theta) - i\sin(n\theta)\\ &= 2\cos(n\theta)\end{aligned}$$

ii. (3 marks)

✓ [1] Obtaining an expression in terms of powers of $\cos\theta$ only, or equivalent merit.

✓ [2] Further progress to solution, including setting up an equation in terms of z, i.e. $z + \frac{1}{z} = 1$ and $z + \frac{1}{z} = -\frac{2}{3}$.

✓ [3] For fully correct solution.

$$3z^2 - z^3 + 4z^2 - z + 3 = 0$$

Dividing throughout by z^2,

$$\begin{aligned} 3z^2 - z + 4 - \frac{1}{z} + \frac{3}{z^2} &= 0 \\ 3\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 4 &= 0 \end{aligned}$$

As $z^n + \frac{1}{z^n} = 2\cos n\theta$,

$$\begin{aligned} 3\left(2\cos 2\theta\right) - 2\cos\theta + 4 &= 0 \\ 6\left(2\cos^2\theta - 1\right) - 2\cos\theta + 4 &= 0 \\ 6\cos^2\theta - 2\cos\theta - 2 &= 0 \end{aligned}$$

Letting $u = \cos\theta$,

$$\begin{aligned} 6u^2 - u - 1 &= 0 \\ (2u-1)(3u+1) &= 0 \\ \therefore u &= \frac{1}{2} \text{ or } -\frac{1}{3} \\ \therefore \cos\theta &= \frac{1}{2} \text{ or } -\frac{1}{3} \end{aligned}$$

As $z + \frac{1}{z} = 2\cos\theta$,

- If $\cos\theta = \frac{1}{2}$

$$\begin{aligned} z + \frac{1}{z} &= 2 \times \frac{1}{2} = 1 \\ z + \frac{1}{z} &= 1 \\ z^2 - z + 1 &= 0 \end{aligned}$$

Applying the quadratic formula,

$$\begin{aligned} z &= \frac{1 \pm \sqrt{1 - 4(1)}}{2} \\ &= \frac{1 \pm \sqrt{3}i}{2} \end{aligned}$$

- If $\cos\theta = -\frac{1}{3}$

$$\begin{aligned} z + \frac{1}{z} &= 2 \times \frac{-1}{3} = -\frac{2}{3} \\ z + \frac{1}{z} &= -\frac{2}{3} \\ 3z^2 + 2z + 3 &= 0 \end{aligned}$$

Applying the quadratic formula,

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{4 - 4(9)}}{2(3)} \\ &= \frac{-2 \pm \sqrt{32}i}{2(3)} \\ &= \frac{-2 \pm 4i\sqrt{2}}{2(3)} \\ &= \frac{-1 \pm 2i\sqrt{2}}{3} \end{aligned}$$

Hence $z = \dfrac{1 \pm \sqrt{3}i}{2}$ or $\dfrac{-1 \pm 2i\sqrt{2}}{3}$.

(d) i. (2 marks)

- ✓ [1] Uses the Binomial theorem to successfully expand $(\cos\theta + i\sin\theta)^5$.
- ✓ [2] For fully correct solution.

$$\begin{aligned} &\cos 5\theta + i\sin 5\theta \\ &= (\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta \\ &\quad + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta \\ &\quad + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta \end{aligned}$$

Equating real parts,

$$\begin{aligned} \cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta \\ &\quad + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta\left(1 - \cos^2\theta\right) \\ &\quad + 5\cos\theta\left(1 - \cos^2\theta\right)^2 \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta \\ &\quad + 5\cos\theta\left(1 - 2\cos^2\theta + \cos^4\theta\right) \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta \\ &\quad + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \end{aligned}$$

ii. (3 marks)

✓ [1] Makes connection between the polynomial equation and the powers of $\cos\theta$.

✓ [2] Finds all values of θ.

✓ [3] For fully correct solution. Does not have to be in lowest multiples of $\frac{\pi}{15}$.

Multiply previous expression by 2:

$$\begin{aligned}&2\left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right)\\&= 32\cos^5\theta - 40\cos^3\theta + 10\cos\theta\end{aligned}$$

Notice that $32x^5 - 40x^3 + 10x - 1 = 0$ is created by letting $x = \cos\theta$, and then equating the entire expression with 1, i.e.

$$\begin{gathered}32\cos^5\theta - 40\cos^3\theta + 10\cos\theta = 1\\ \underbrace{32\cos^5\theta - 40\cos^3\theta + 10\cos\theta}_{=2\cos 5\theta} - 1 = 0\end{gathered}$$

Hence the polynomial equation can be solved by solving

$$\begin{aligned}2\cos 5\theta - 1 &= 0\\ \cos 5\theta &= \frac{1}{2}\\ 5\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}\\ &\quad \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}, \frac{23\pi}{3},\\ &\quad \frac{25\pi}{3}, \frac{29\pi}{3}\\ \therefore \theta &= \frac{\pi}{15}, \frac{5\pi}{15}, \frac{7\pi}{15}, \frac{11\pi}{15}\\ &\quad \frac{13\pi}{15}, \frac{17\pi}{15}, \frac{19\pi}{15}, \frac{23\pi}{15},\\ &\quad \frac{25\pi}{15}, \frac{29\pi}{15}\end{aligned}$$

Hence solutions to the polynomial are:

$$\begin{aligned}x &= \cos\frac{\pi}{15}, \underbrace{\cos\frac{\pi}{3}}_{=\frac{1}{2}}, \cos\frac{7\pi}{15}, \cos\frac{11\pi}{15}\\ &\cos\frac{13\pi}{15}, \underbrace{\cos\frac{17\pi}{15}}_{=\cos\frac{13\pi}{15}}, \underbrace{\cos\frac{19\pi}{15}}_{=\cos\frac{11\pi}{15}}, \underbrace{\cos\frac{23\pi}{15}}_{=\cos\frac{7\pi}{15}},\\ &\underbrace{\cos\frac{5\pi}{3}}_{=\cos\frac{\pi}{3}}, \underbrace{\cos\frac{29\pi}{15}}_{=\cos\frac{\pi}{15}}\end{aligned}$$

However, there are several roots which double up, and the power 5 polynomial will only contain 5 unique solutions, which are denoted by the roots that have equivalent values stated.

$$\begin{aligned}\therefore x = \cos\frac{\pi}{15}, \frac{1}{2}, \cos\frac{7\pi}{15}\\ \underbrace{\cos\frac{11\pi}{15}}_{=-\cos\frac{4\pi}{15}}, \underbrace{\cos\frac{13\pi}{15}}_{=-\cos\frac{2\pi}{15}}\end{aligned}$$

iii. (2 marks)

✓ [1] Uses the product of roots and equate to $\frac{1}{32}$, or equivalent.

✓ [2] For fully correct solution.

Multiplying the roots to the power 5 polynomial,

$$\begin{aligned}&\left(\cos\frac{\pi}{15}\right)\frac{1}{2}\left(-\cos\frac{2\pi}{15}\right)\left(-\cos\frac{4\pi}{15}\right)\left(\cos\frac{7\pi}{15}\right)\\ &= -\frac{f}{a} = \frac{1}{32}\\ &\therefore \cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{16}\end{aligned}$$

Question 15 (Ham)

(a) i. (3 marks)

✓ [1] Sets up contradiction that involves $\gcd(p,q) = 1$

✓ [2] Substantial progress to show that $14q^2 = p^2$, i.e. both sides are even.

✓ [3] Fully correct solution.

Assume $\sqrt{14}$ is rational. Then,

$$\exists p, q \in \mathbb{Z}^+ \text{ s.t. } \sqrt{14} = \frac{p}{q}$$

with $\gcd(p,q) = 1$. Squaring both sides,

$$\begin{aligned}14 &= \frac{p^2}{q^2}\\ \therefore 14q^2 &= p^2\end{aligned}$$

Hence p is even. Rewriting: $p = 2k$, where $k \in \mathbb{Z}^+$,

$$\begin{aligned} 14q^2 &= (2k)^2 \\ 14q^2 &= 2 \times 2k \\ 7q^2 &= 2k \end{aligned}$$

Which implies $7q^2$ is even. The only possibility is q^2 is even.

But this is not possible as the original assumption was that $\gcd(p, q) = 1$, so p and q cannot be both even, which is a contradiction. Hence $\sqrt{14}$ is not rational.

ii. (2 marks)

✓ [1] Makes mathematical reference to $\sqrt{7n}$ and $\sqrt{2n}$ being both rational, i.e. $\sqrt{7n} = \frac{a}{b}$ and $\sqrt{2n} = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}^+$

✓ [2] for fully correct solution.

Suppose $\sqrt{7n}$ and $\sqrt{2n}$ are both rational for some $a, b, c, d \in \mathbb{Z}^+$ such that

$$\sqrt{7n} = \frac{a}{b} \qquad \sqrt{2n} = \frac{c}{d}$$

Multiplying,

$$\begin{aligned} \sqrt{7n} \times \sqrt{2n} &= \frac{ac}{bd} \\ \sqrt{14n^2} &= \frac{ac}{bd} \\ \sqrt{14}n &= \frac{ac}{bd} \\ \sqrt{14} &= \frac{ac}{nbd} \end{aligned}$$

which implies $\sqrt{14}$ is rational as $a, b, c, d, n \in \mathbb{Z}^+$. However, $\sqrt{14}$ is clearly not rational from part (i). Hence a contradiction exists and $\sqrt{7n}$ and $\sqrt{2n}$ cannot be both rational.

(b) i. (4 marks)

✓ [1] Correctly tests the base case.

✓ [2] Substantial progress to correctly incorporate $P(k)$ into $P(k + 1)$.

✓ [3] Substantial progress to entirely remove fractions within fractions.

✓ [4] For fully correct solution.

Let $P(n)$ be the proposition

$$a_n = 2\left(\frac{1 + \left(-\frac{1}{3}\right)^n}{1 - \left(-\frac{1}{3}\right)^n}\right)$$

with $a_1 = 1$ and $a_{n+1} = \dfrac{4 + a_n}{1 + a_n}$

- Base case: $P(1)$

$$\begin{aligned} a_1 &= 2\left(\frac{1 - \frac{1}{3}}{1 + \frac{1}{3}}\right) = 2\left(\frac{\frac{2}{\cancel{3}}}{\frac{4}{\cancel{3}}}\right) \\ &= 2 \times \frac{1}{2} = 1 \end{aligned}$$

Hence $P(1)$ is true.

- Inductive hypothesis: assume $P(k)$ is true, i.e.

$$a_k = 2\left(\frac{1 + \left(-\frac{1}{3}\right)^k}{1 - \left(-\frac{1}{3}\right)^k}\right)$$

with $a_1 = 1$ and $a_{k+1} = \dfrac{4 + a_k}{1 + a_k}$

Examine $P(k + 1)$:

$$\text{RTP: } a_{k+1} = 2\left(\frac{1 + \left(-\frac{1}{3}\right)^{k+1}}{1 - \left(-\frac{1}{3}\right)^{k+1}}\right)$$

$$\begin{aligned} &a_{k+1} \\ &= \frac{4 + a_k}{1 + a_k} \\ &= \frac{4 + 2\left(\frac{1+\left(-\frac{1}{3}\right)^k}{1-\left(-\frac{1}{3}\right)^k}\right)}{1 + 2\left(\frac{1+\left(-\frac{1}{3}\right)^k}{1-\left(-\frac{1}{3}\right)^k}\right)} \frac{\times\left(1-\left(-\frac{1}{3}\right)^k\right)}{\times\left(1-\left(-\frac{1}{3}\right)^k\right)} \\ &= \frac{4\left(1 - \left(-\frac{1}{3}\right)^k\right) + 2\left(1 + \left(-\frac{1}{3}\right)^k\right)}{\left(1 - \left(-\frac{1}{3}\right)^k\right) + 2\left(1 + \left(-\frac{1}{3}\right)^k\right)} \\ &= \frac{6 + \left(-\frac{1}{3}\right)^k(-4 + 2)}{3 + \left(-\frac{1}{3}\right)^k(2 - 1)} \\ &= 2\left(\frac{3 - \left(-\frac{1}{3}\right)^k}{3 + \left(-\frac{1}{3}\right)^k}\right) \\ &= 2\left(\frac{\cancel{3}\left(1 + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^k\right)}{\cancel{3}\left(1 - \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^k\right)}\right) \\ &= 2\left(\frac{1 + \left(-\frac{1}{3}\right)^{k+1}}{1 - \left(-\frac{1}{3}\right)^{k+1}}\right) \end{aligned}$$

Hence $P(k + 1)$ is also true, and $P(n)$ is true by induction.

ii. (1 mark)

$$\lim_{n\to\infty}\left(-\frac{1}{3}\right)^n = 0$$

Hence

$$\lim_{n\to\infty} a_n = 2$$

(c) i. (3 marks)

✓ [1] Demonstrates repeated use of the result provided to show

$$x^2+y^2+z^2 \geq xy+xz+yz$$

or equivalent merit.

✓ [2] Successfully factorises

$$x^3+y^3+z^3-3xyz$$

or equivalent merit.

✓ [3] For fully correct solution.

Given $\dfrac{x+y}{2} \geq \sqrt{xy}$, replacing x with x^2 etc,

$$\frac{x^2+y^2}{2} \geq xy \qquad (15.1)$$

Repeating for xz and yz:

$$\frac{x^2+z^2}{2} \geq xz \qquad (15.2)$$

$$\frac{x^2+z^2}{2} \geq yz \qquad (15.3)$$

Adding (15.1), (15.2) and (15.3),

$$x^2+y^2+z^2 \geq xy+xz+yz \qquad (15.4)$$

Now check to see that $(x+y+z)$ is a factor of $x^3+y^3+z^3-3xyz$ by using the factor theorem with

$$f(x) = x^3+y^3+z^3-3xyz$$

and if $f(-y-z)=0$, then $x=-y-z$ would be a zero.

Note: Cambridge Year 11 Ext 1, Ex 10D Q18

$$\begin{aligned}
&f(-y-z)\\
&= (-y-z)^3+y^3+z^3-3xyz\\
&= -\left(y^3+3y^2z+3yz^2+z^3\right)\\
&\quad +y^3+z^3-3(-y-z)yz\\
&= -y^3-3y^2z-3yz^2-z^3+y^3+z^3\\
&\quad +3y^2z+3yz^2\\
&= 0
\end{aligned}$$

Hence $x+y+z$ is a factor of $f(x)$. Factorising,

$$\begin{aligned}
&x^3+y^3+z^3-3xyz\\
&= (x{+}y{+}z)\left(x^2+y^2+z^2-xy-xz-yz\right)
\end{aligned} \qquad (15.5)$$

Now as a result from (15.4),

$$x^2+y^2+z^2-xy-xz-yz \geq 0$$

and given x, y and z are positive real numbers, then

$$(x{+}y{+}z)\left(x^2+y^2+z^2-xy-xz-yz\right) \geq 0$$

and hence

$$\begin{aligned}
x^3+y^3+z^3-3xyz &\geq 0\\
x^3+y^3+z^3 &\geq 3xyz
\end{aligned}$$

Replace x with $a^{\frac{1}{3}}$, y with $b^{\frac{1}{3}}$ and z with $c^{\frac{1}{3}}$,

$$a+b+c \geq 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$$

Take to power 3 on both sides,

$$(a+b+c)^3 \geq 27abc$$

ii. (2 marks)

✓ [1] Replaces a, b and c with the required values.

✓ [2] Fully correct solution

From the previous part, and given $abc=1$, then

$$(a+b+c)^3 \geq 27$$

Replace a with $\frac{a}{b^4}$, b with $\frac{b}{c^4}$ and c with $\frac{c}{a^4}$, and cube rooting both sides,

$$\begin{aligned}
\frac{a}{b^4}+\frac{b}{c^4}+\frac{c}{a^4} &\geq 3\\
\frac{a\left(a^4c^4\right)+b\left(a^4b^4\right)+c\left(b^4c^4\right)}{\cancelto{1}{a^4b^4c^4}} &\geq 3\\
\therefore a^5c^4+b^5a^4+c^5b^4 &\geq 3
\end{aligned}$$

or replace a with a^5c^4, b with b^5a^4 and c with c^5b^4,

$$\begin{aligned}
(a^5c^4+b^5a^4+c^5b^4)^3 &\geq 27(abc)^9\\
&= 27\\
&(\because abc=1)\\
\therefore a^5c^4+b^5a^4+c^5b^4 &\geq 3
\end{aligned}$$

Question 16 (Ham)

(a) (3 marks)

- ✓ [1] Find the expressions for $\frac{1}{2}v^2$ in terms of x, or equivalent merit. **Note:** terminating error if integrating w.r.t. t instead of x.
- ✓ [2] Substantial progress to find the expression for v in terms of x, including choosing the correct root.
- ✓ [3] For fully correct solution.

$$\begin{aligned}\ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x} \\ \frac{1}{2}v^2 &= -2\int e^{-x}\,dx = 2e^{-x} + C_1\end{aligned}$$

$t = 0$, $x = 0$, $v = 2$:

$$\begin{aligned}\frac{1}{2}(4) &= 2e^0 + C_1 \\ 2 &= 2 + C_1 \\ C_1 &= 0 \\ \therefore \frac{1}{2}v^2 &= 2e^{-x} \\ v^2 &= 4e^{-x} \\ v &= \pm 2e^{-\frac{1}{2}x}\end{aligned}$$

$t = 0$, $x = 0$ and $v = 2 > 0$ and as $v \neq 0$, it indicates that the particle only travels in one direction

$$\begin{aligned}\therefore 2 &= +2e^0 \\ \therefore v &= 2e^{-\frac{1}{2}x}\end{aligned}$$

Rewriting v as $\frac{dx}{dt}$:

$$\begin{aligned}\frac{dx}{dt} &= 2e^{-\frac{1}{2}x} \\ e^{\frac{1}{2}x}\,dx &= 2\,dt\end{aligned}$$

Integrating,

$$\begin{aligned}\int e^{\frac{1}{2}x}\,dx &= \int 2\,dt \\ 2e^{\frac{1}{2}x} &= 2t + C_1\end{aligned}$$

When $t = 0$, $x = 0$

$$\begin{aligned}2e^0 &= 2(0) + C_1 \\ \therefore C_1 &= 2 \\ 2e^{\frac{1}{2}x} &= 2t + 2 \\ e^{\frac{1}{2}x} &= t + 1 \\ \frac{1}{2}x &= \log_e(t+1) \\ x &= 2\log_e(t+1)\end{aligned}$$

(b) i. (3 marks)

- ✓ [1] Attempts to use integration by parts.
- ✓ [2] Makes substantial progress towards the solution with only minor errors.
- ✓ [3] For fully correct solution.

$$\text{Let } I = \int e^{-x}\sin x\,dx$$

$$\left|\begin{array}{ll} u = e^{-x} & v = -\cos x \\ du = -e^{-x} & dv = \sin x \end{array}\right.$$

$$\begin{aligned}I &= \Big[uv\Big] - \int v\,du \\ &= -e^{-x}\cos x - \int e^{-x}\cos x\,dx\end{aligned}$$

Applying integration by parts again,

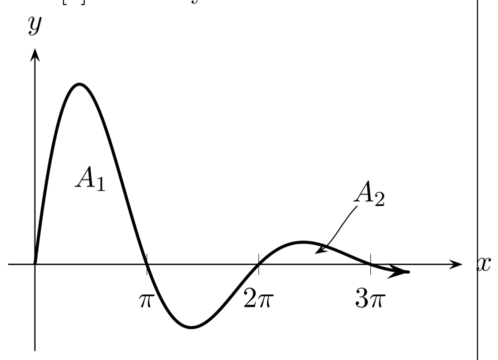
$$\left|\begin{array}{ll} u = e^{-x} & v = \sin x \\ du = -e^{-x} & dv = \cos x \end{array}\right.$$

$$\begin{aligned}&\int e^{-x}\cos x\,dx \\ &= e^{-x}\sin x + \int e^{-x}\sin x\,dx \\ &= e^{-x}\sin x + I \\ \therefore I &= -e^{-x}\cos x - \left(e^{-x}\sin x + I\right) + C \\ &= -e^{-x}\sin x - e^{-x}\cos x - I + C \\ \therefore 2I &= -e^{-x}(\sin x + \cos x) + C \\ I &= -\frac{1}{2}e^{-x}(\sin x + \cos x) + C\end{aligned}$$

ii. (2 marks)

✓ [1] Correctly finds the primitive for A_n.

✓ [2] For fully correct solution



$$A_n = \int_{(2n-2)\pi}^{(2n-1)\pi} e^{-x} \sin x \, dx$$
$$= -\frac{1}{2}\Big[e^{-x}(\cos x + \sin x)\Big]_{(2n-2)\pi}^{(2n-1)\pi}$$

Now $(2n-1)\pi$ is an odd multiple of π, with

- $\cos(2n-1)\pi = -1$
- $\sin(2n-1)\pi = 0$

Similarly, $(2n-2)\pi$ is an even multiple of π, with

- $\cos(2n-2)\pi = 1$
- $\sin(2n-2)\pi = 0$

$$\begin{aligned} A_n &= -\frac{1}{2}\Big(e^{-(2n-1)\pi}(-1) \\ &\quad - e^{-(2n-2)\pi}(1)\Big) \\ &= \frac{1}{2}\left(e^{(1-2n)\pi} + e^{(2-2n)\pi}\right) \end{aligned}$$

(c) (3 marks)

✓ [1] Finding all of A_1, A_2 and A_3 correctly.

✓ [2] For finding the parameters of the limiting sum with reasoning.

✓ [3] For fully correct solution.

$$\begin{aligned} A_1 &= \frac{1}{2}\left(e^{(1-2)\pi} + e^0\right) \\ &= \frac{1}{2}\left(e^{-\pi} + 1\right) \\ A_2 &= \frac{1}{2}\left(e^{-3\pi} + e^{-2\pi}\right) \\ A_3 &= \frac{1}{2}\left(e^{-5\pi} + e^{-4\pi}\right) \end{aligned}$$

Adding the A terms,

$$\begin{aligned} &A_1 + A_2 + A_3 + \cdots \\ &= \frac{1}{2}\big(1 + e^{-\pi} + e^{-2\pi} \\ &\quad + e^{-3\pi} + e^{-4\pi} + e^{-5\pi} + \cdots\big) \end{aligned}$$

This forms a geometric series with $a = 1$, $r = e^{-\pi}$, such that $|r| < 1$:

$$S = \frac{1}{1 - e^{-\pi}} \frac{\times e^{\pi}}{\times e^{\pi}} = \frac{e^{\pi}}{e^{\pi} - 1}$$
$$\therefore A_1 + A_2 + A_3 + \cdots = \frac{e^{\pi}}{2\left(e^{\pi} - 1\right)}$$

(d) (2 marks)

✓ [1] Finds the areas below the x axis.

✓ [2] For fully correct solution.

- Let $X = A_1 + A_2 + A_3 + \cdots$ be the areas above the x axis.
- Let $Y = a_1 + a_2 + a_3 + \cdots$ be the areas below the x axis.
- Given $\displaystyle\lim_{n\to\infty} \int_0^n e^{-x} \sin x \, dx$, this is a subtraction of the areas Y from X, i.e.

$$\frac{1}{2} = \lim_{n\to\infty} \int_0^n e^{-x} \sin x \, dx = X - Y$$

$$\begin{aligned} \frac{1}{2} &= \frac{e^{\pi}}{2\left(e^{\pi} - 1\right)} - Y \\ Y &= \frac{e^{\pi}}{2\left(e^{\pi} - 1\right)} - \frac{1}{2} \\ &= \frac{e^{\pi} - \left(e^{\pi} - 1\right)}{2\left(e^{\pi} - 1\right)} \\ &= \frac{1}{2\left(e^{\pi} - 1\right)} \end{aligned}$$

$$\begin{aligned} &\therefore \lim_{n\to\infty} \int_0^n \left|e^{-x} \sin x\right| \, dx \\ &= X + Y \\ &= \frac{e^{\pi}}{2\left(e^{\pi} - 1\right)} + \frac{1}{2\left(e^{\pi} - 1\right)} \\ &= \frac{e^{\pi} + 1}{2\left(e^{\pi} - 1\right)} \end{aligned}$$